Canard Behavior in Rate Induced Tipping

Jonathan Hahn, November 1, 2016

Rate Induced Tipping

General framework:

$$\frac{df}{dt} = f(x, \mu, \lambda(rt))$$

x is the state vector, μ is a vector of parameters, λ is a continuous function, r is the rate of the forcing

For all values of λ, there is a stable equilibrium $\tilde{x}(\lambda)$

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Rate Induced Tipping

For $r \in [0, r_c)$, λ changes slowly enough that if $x(0)$ is within some neighborhood of $\tilde{x}(\lambda(0))$, then $x(t)$ is within some other neighborhood of $\tilde{x}(\lambda(rt))$ for all t.

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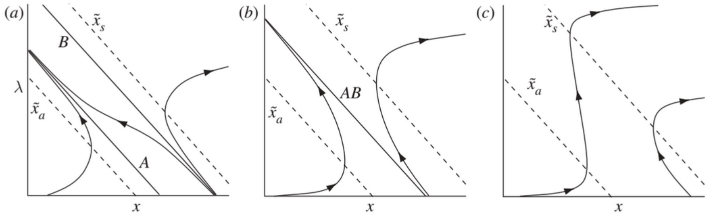
Rate Induced Tipping

For $r > r_c$, $x(t)$ no longer stays within the required neighborhood of the QSE. The system "tips" and we say it has *rate-dependent tipping*.

The neighborhood of the QSE that describes the tipping point can be chosen in many ways: by a given distance R, by the state leaving a basin of attraction of the QSE, by something else topological in the system, or by some other arbitrary choice.

Rate-induced tipping example

$$\frac{dx}{dt} = (x+\lambda)^2 - \mu$$
$$\frac{d\lambda}{dt} = r$$



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Co-moving system: set $w = x + \lambda$.

$$\frac{dw}{dt} = w^2 - \mu + r$$

Equilibrium at $w = \pm\sqrt{\mu - r}$ if $r < \mu$.

Tipping condition: $r > r_c$ where

$$r_c = \begin{cases} \mu - (\lambda_0 + x_0)^2 & \textit{if } -x_0 < \lambda_0 < -x_0 + \sqrt{\mu} \\ \mu & \textit{if } \lambda_0 \leq -x_0 \end{cases}$$

Fast-Slow System

QSE near a locally folded critical manifold.

$$\epsilon \frac{dx}{dt} = y + \lambda + x(x-1)$$
$$\frac{dy}{dt} = -\sum_{n=1}^{N} x^n$$

$N \geq 5$, odd.
$(0, -\lambda)$ is globally asymptotically stable.

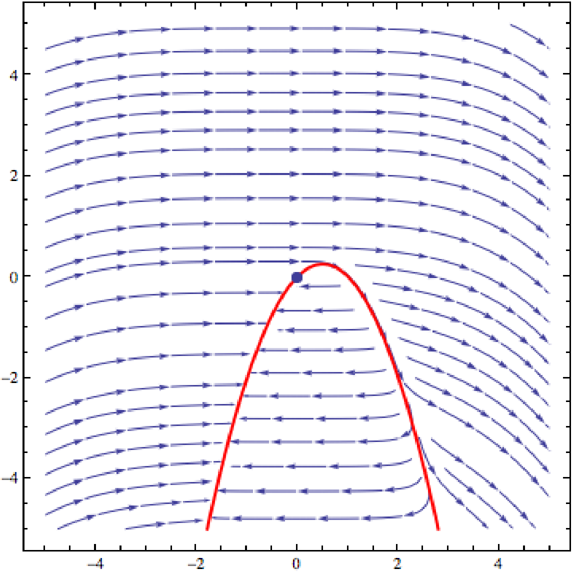
Fast-Slow System

$$\epsilon \frac{dx}{dt} = y + \lambda + x(x-1)$$
$$\frac{dy}{dt} = -\sum_{n=1}^{N} x^n$$

Set $\epsilon = 0$ to find the slow manifold: $0 = y + \lambda + x(x-1)$

$S(\lambda) = \{(x, y) \in \mathbb{R}^2 : y = -\lambda - x(x-1)\}$

Fast-Slow System



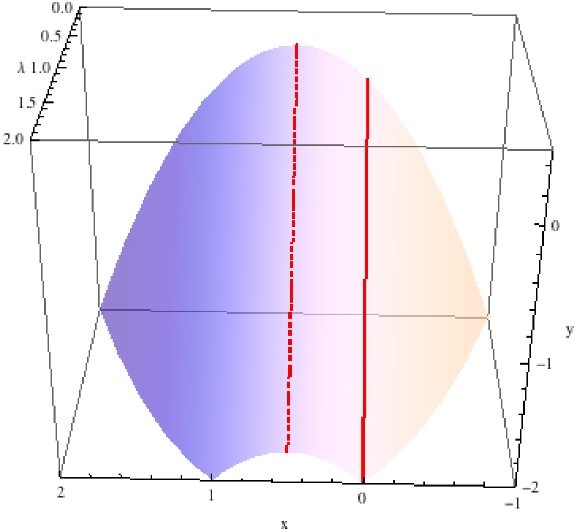
Fast Slow System: Rate Tipping

$$\epsilon \frac{dx}{dt} = y + \lambda + x(x-1)$$

$$\frac{dy}{dt} = -\sum_{n=1}^{N} x^n$$

$$\frac{d\lambda}{dt} = r$$

Critical Manifold



Projected Reduced System

Set $\epsilon = 0$ and differentiate the resulting equation with respect to t to find a system approximating the slow dynamics.

$$0 = \frac{dy}{dt} + \frac{d\lambda}{dt} + (2x - 1)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(\sum_{n=1}^{N} x^n - r\right)(2x - 1)^{-1}$$

$$\frac{d\lambda}{dt} = r$$

Desingularized system

Rescale time: $\frac{dt}{d\tau} = -(2x - 1)$

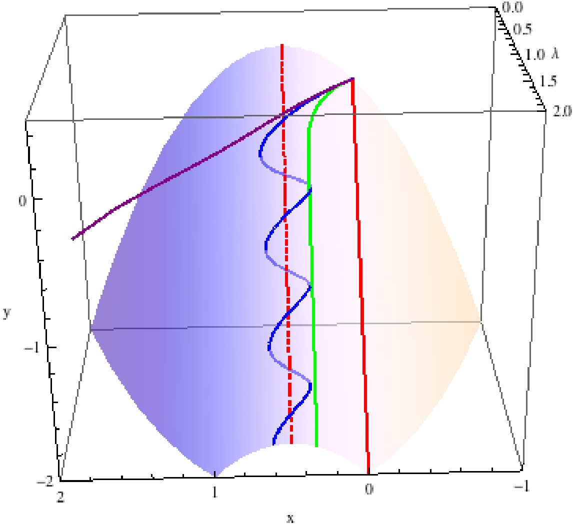
$$\frac{dx}{d\tau} = \left(r - \sum_{n=1}^{N} x^n\right)$$
$$\frac{d\lambda}{d\tau} = -r(2x - 1)$$

This reverses the direction of time on the repelling part of the critical manifold.
For $0 < r < \sum_{n=1}^{N}(1/2)^n$, all trajectories within the attracting part of the critical manifold converge to x^* where $r = \sum_{n=1}^{N} x^*$.

Critical rate

$$r_c = \sum_{n=1}^{N} (1/2)^n$$

Solutions



Co-moving system

$$\epsilon \frac{dx}{dt} = y + \lambda + x(x-1)$$
$$\frac{dy}{dt} = -\sum_{n=1}^{N} x^n$$
$$\frac{d\lambda}{dt} = r$$

Create a co-moving system: $w = y + \lambda$

$$\epsilon \frac{dx}{dt} = w + x(x-1)$$
$$\frac{dw}{dt} = r - \sum_{n=1}^{N} x^n$$

Co-moving system

$$\epsilon \frac{dx}{dt} = w + x(x-1)$$
$$\frac{dw}{dt} = r - \sum_{n=1}^{N} x^n$$

The equilibrium (x^*, w^*) in the co-moving system is given by the solution to

$$\sum_{n=1}^{N} (x^*)^n = r$$
$$w^* = -(x^*)^2 + x^*$$

Hopf bifurcation analysis

The Jacobian at this equilibrium is:

$$\begin{bmatrix} (2x_1^* - 1)/\epsilon & 1/\epsilon \\ \sum_{n=1}^{N} -n(x_1^*)^{n-1} & 0 \end{bmatrix}$$

and the eigenvalues of the Jacobian are

$$\frac{2x_1^* - 1 \pm \sqrt{(1-2x_1^*)^2 - 4\epsilon \sum_{n=1}^{N} n(x_1^*)^{n-1}}}{2\epsilon}.$$

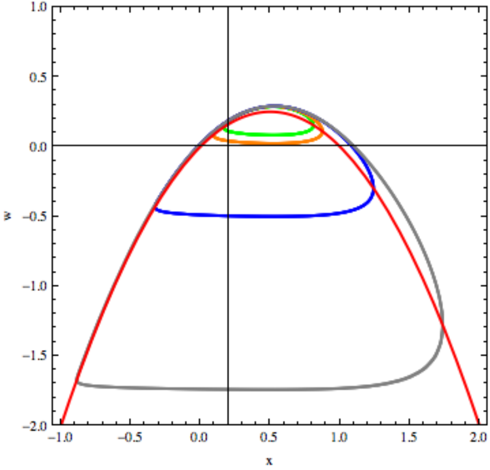
Hopf bifurcation analysis

Eigenvalues:

$$\frac{2x_1^* - 1 \pm \sqrt{(1-2x_1^*)^2 - 4\epsilon \sum_{n=1}^{N} n(x_1^*)^{n-1}}}{2\epsilon}.$$

When $x^* < 1/2$, the equilibrium is stable, which is in agreement with the previous conclusion that the system does not tip for $r < \sum_{n=1}^{N}(1/2)^n = r_c$. As r increases, so does x^*, so when $r = r_c$, and $x^* = 1/2$ the pair of eigenvalues cross the imaginary axis, and a Hopf bifurcation occurs.

Periodic orbits in co-moving system



Periodic orbit expands rapidly as in a canard explosion.

Van der Pol Oscillator

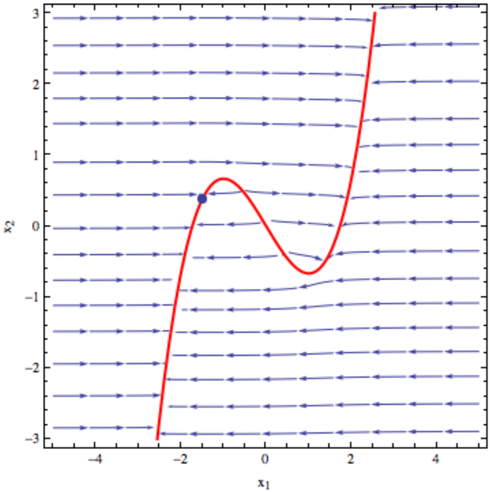
$$\epsilon \frac{dx}{dt} = x_2 + (x_1 - \frac{x_1^3}{3})$$
$$\frac{dx_2}{dt} = -x_1 - \alpha$$

Van der Pol Oscillator

$$\epsilon\frac{dx}{dt} = x_2 + (x_1 - \frac{x_1^3}{3}) + \lambda$$
$$\frac{dx_2}{dt} = -x_1 - \alpha$$
$$\frac{d\lambda}{dt} = r > 0$$

$\alpha > 1$ is fixed, and a stable equilibrium exists at $(-\alpha, -\alpha + \frac{\alpha^3}{3} - \lambda)$

Phase portrait for $\lambda = 0$.



Co-moving van der Pol System

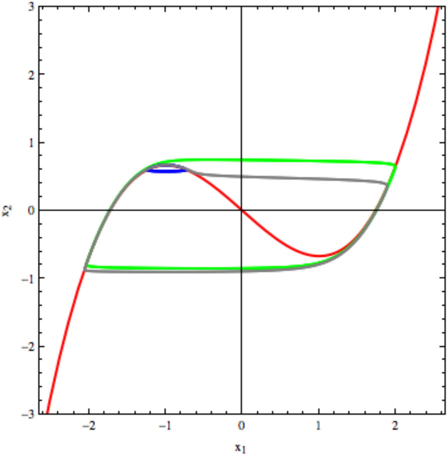
Set $w = x_2 + \lambda$:

$$\epsilon \frac{dx_1}{dt} = w + (x_1 - \frac{x_1^3}{3})$$
$$\frac{dw}{dt} = -x_1 - \alpha + r$$

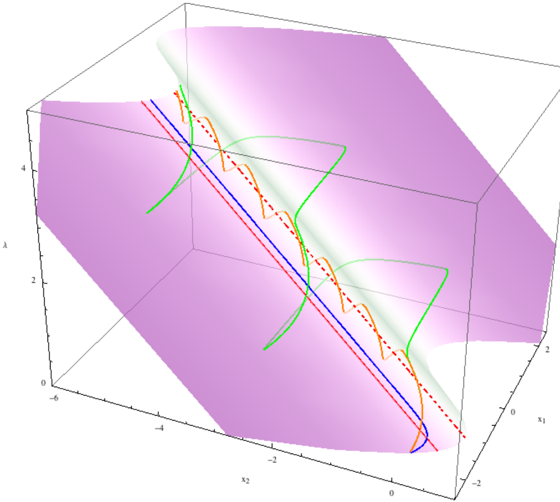
This is the classic van der Pol system!

Canard Progression

As r increases beyond $\alpha - 1$, there is a canard explosion.



Orbits in original system



Questions

Does spiraling behavior still count as "tracking"?

If so, is the critical rate for spiraling really a "tipping point"?

How do we prove this sprialing occurs when we can't reduce to a "co-moving system"?